

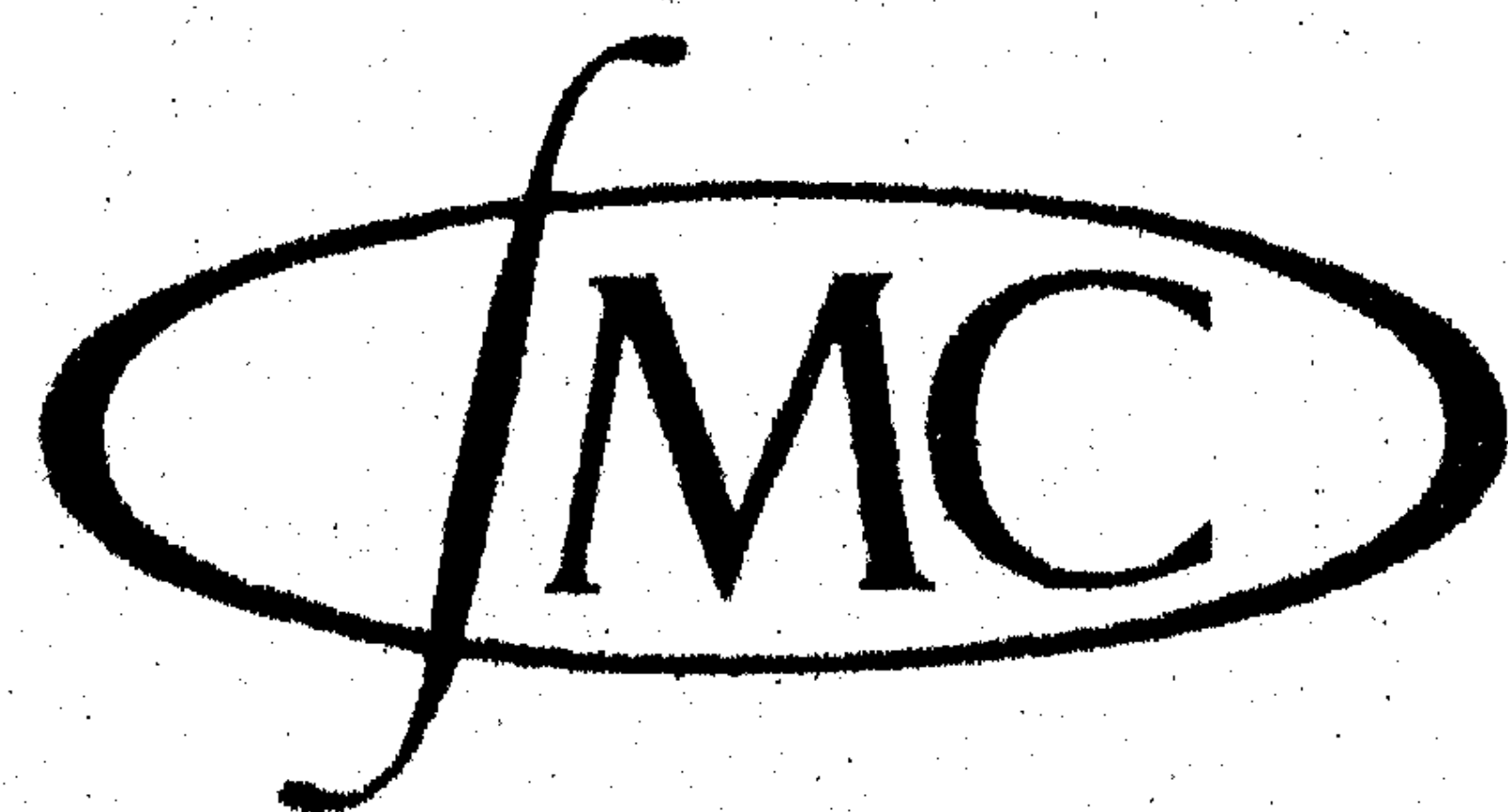
STICHTING
MATHEMATISCH CENTRUM
2e BOERHAAVESTRAAT 49
AMSTERDAM

ZW 1967-11r

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Reprinted from:
General topology and its relations to modern
analysis and algebra, 2. Prague 1966, p 161-163.



1967

R E P R I N T

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to Modern Analysis and Algebra II**

**Proceedings of the Second
Prague Topological Symposium, 1966**

THE COMPACTNESS OPERATOR IN GENERAL TOPOLOGY

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The role of (bi)compactness has increased tremendously during the last half century. This abstract indicates a further strengthening of this notion (at the expense of the Hausdorff property, e.g.).

Let X be a set, and \mathcal{F} a family of subsets of X . Let ε denote the operator, which assigns to \mathcal{F} the collection $\varepsilon(\mathcal{F})$, that is the family of all finite unions and arbitrary intersections of members from \mathcal{F} . We do not assume that $\varepsilon(\mathcal{F})$ necessarily contains \emptyset and X as elements.

Such a family $\varepsilon(\mathcal{F})$ on a set X is called a minus-topology $(X, \varepsilon(\mathcal{F}))$ over X . (It can, of course, always be extended to a topology over X .)

A subset S of X is called compact relative to \mathcal{F} , as usual, provided that every subfamily \mathcal{F}' of \mathcal{F} , for which $\mathcal{F}' \cup \{S\}$ has the finite intersection property, has a non empty intersection in S . So, to any \mathcal{F} corresponds a family of compact sets $\varrho(\mathcal{F})$ in $(X, \varepsilon(\mathcal{F}))$, where ϱ is called the compactness operator.

The elements of $\varrho(\varrho(\mathcal{F})) = \varrho^2(\mathcal{F})$ are called square-compact subsets of $(X, \varepsilon(\mathcal{F}))$. A subset S of X is apparently square-compact, if every subcollection $(\varrho(\mathcal{F}))'$ of $\varrho(\mathcal{F})$ for which $(\varrho(\mathcal{F}))' \cup \{S\}$ has the finite intersection property, has a non empty intersection in S . We call $\varrho^2 = \sigma$ the square-compactness operator.

We have the following connections between these operators.

$$(1) \quad \varrho\varepsilon = \varrho.$$

Observe that (1) is a reformulation of Alexander's Lemma!

$$(2) \quad \varepsilon\sigma = \sigma\varepsilon = \sigma;$$

$$(3) \quad \varepsilon^2 = \varepsilon; \quad \sigma^2 = \sigma.$$

For the proof of the propositions (2) and (3) we need a lemma.

Lemma. *Let C be a subset of X and an element of $\varrho(\mathcal{F})$; let E be a subset of X and an element of $\varrho^2(\mathcal{F})$. Then $C \cap E$ is an element of $\varrho^2(\mathcal{F}) \cap \varrho(\mathcal{F})$.*

Proof. a) Let \mathcal{C}' be a sub-collection of $\varrho(\mathcal{F})$ such that $\mathcal{C}' \cup \{C \cap E\}$ has the finite intersection property. Then $\mathcal{C}' \cup \{C\} \cup \{E\}$ has the finite intersection property

(further written f.i.p.). But since $\mathcal{C}' \cup \{C\} \subset \varrho(\mathcal{F})$ and $E \in \varrho^2(\mathcal{F})$ we have $(\bigcap \mathcal{C}') \cap C \cap E \neq \emptyset$ which proves that $C \cap E \in \varrho^2(\mathcal{F})$.

b) Choose $\mathcal{F}' \subset \mathcal{F}$ such that $\mathcal{F}' \cup \{C \cap E\}$ has the finite intersection property. Then the collection $\mathcal{F}'' = \{F \cap C \mid F \in \mathcal{F}'\}$ has also the finite intersection property in E .

It is obvious that the elements of \mathcal{F}'' are compact relative to \mathcal{F} , because each element is an intersection of a subbasic closed set and a compact set. Hence \mathcal{F}'' is a subcollection of $\varrho(\mathcal{F})$ with the finite intersection property in E and consequently $(\bigcap \mathcal{F}'') \cap E$ is non empty. From this we obtain $(\bigcap \mathcal{F}') \cap (C \cap E) \neq \emptyset$, thus $(C \cap E) \in \varrho(\mathcal{F})$.

Proposition (2). *The collection $\varrho^2(\mathcal{F}) = \sigma(\mathcal{F})$ is closed under finite unions and arbitrary intersections.*

Proof. The fact that $\varrho^2(\mathcal{F})$ is closed under finite unions is a consequence of the definition of $\varrho^2(\mathcal{F})$. Now we will prove that $\varrho^2(\mathcal{F})$ is closed under arbitrary intersections.

Consider a collection $\mathcal{E}' \subset \varrho^2(\mathcal{F})$ such that $\bigcap \mathcal{E}' = E_0 \neq \emptyset$, (the case that $\bigcap \mathcal{E}' = \emptyset$ is trivial).

We must prove that every collection \mathcal{C}' , such that $\mathcal{C}' \cup \{E_0\}$ has the f.i.p., has a non empty intersection in E_0 .

Pick and fix a member $E_1 \in \mathcal{E}'$ and consider the collection $\mathcal{C}'' = \{C \cap E \mid C \in \mathcal{C}'; E \in \mathcal{E}'\}$.

From the Lemma it follows that the members of \mathcal{C}'' are members of $\varrho(\mathcal{F})$. By assumption $\mathcal{C}'' \cup \{E_1\}$ has the f.i.p. and hence $(\bigcap \mathcal{C}'') \cap E_1 \neq \emptyset$; but this intersection equals $(\bigcap \mathcal{C}') \cap E_0$ and this proves that $E_0 = (\bigcap \mathcal{E}') \in \varrho^2(\mathcal{F})$.

Proposition (3). $\varrho^2(\mathcal{F}) = \varrho^4(\mathcal{F})$.

Proof. We first prove that $\varrho(\mathcal{F}) \subset \varrho^3(\mathcal{F})$. Let C be an element of $\varrho(\mathcal{F})$ and let \mathcal{E}' be a subcollection of $\varrho^2(\mathcal{F})$ such that $\mathcal{E}' \cup \{C\}$ has the f.i.p.

Pick and fix some $E_0 \in \mathcal{E}'$ and consider $\tilde{\mathcal{C}} = \{C \cap E \mid E \in \mathcal{E}'\}$.

From the Lemma it follows that each member of $\tilde{\mathcal{C}}$ is a member of $\varrho(\mathcal{F})$ and clearly $\tilde{\mathcal{C}} \cup \{E_0\}$ has the f.i.p.

Thus $(\bigcap \tilde{\mathcal{C}}) \cap E_0 \neq \emptyset$; $C \cap (\bigcap \mathcal{E}') \neq \emptyset$ and hence C is a member of $\varrho^3(\mathcal{F})$, which proves that $\varrho(\mathcal{F}) \subset \varrho^3(\mathcal{F})$.

Similarly we can find that $\varrho^2(\mathcal{F}) \subset \varrho^4(\mathcal{F})$.

On the other hand $\varrho^2(\mathcal{F})$ is defined as being the collection of compact sets relative to $\varrho(\mathcal{F})$ and $\varrho^4(\mathcal{F})$ as being the collection of compact sets relative to $\varrho^3(\mathcal{F})$. From $\varrho(\mathcal{F}) \subset \varrho^3(\mathcal{F})$ it follows that $\varrho^2(\mathcal{F}) \supset \varrho^4(\mathcal{F})$.

Hence $\varrho^2(\mathcal{F}) = \varrho^4(\mathcal{F})$.

$\varepsilon\sigma = \sigma$ says that for every \mathcal{F} the family $\varrho^2(\mathcal{F})$ forms a minus topology on X .

The second part of (3) tells us in particular that the ϱ operator is "of finite order" and the relations (2) and (3) determine the structure of the semigroup $\{\varepsilon, \sigma\}$; ε is an identity, and σ is an idempotent.

Let us discuss now a few special cases of importance.

I. $\varrho = \varepsilon$ holds exactly for those topological spaces in which the compact sets coincide with the closed sets. The results above become trivial.

II. $\varrho^2 = \varepsilon$. In this case ϱ and ε form a group of order 2 with ε as the identity. This case has been studied in [1]. Spaces supplied with such a minus topology are called antispaces. These are exactly those spaces in which the square-compact subsets coincide with the closed subsets. The locally compact Hausdorff spaces and the metrizable spaces are e.g. antispaces.

If (X, \mathcal{G}) is an antispaces with a minus topology, then also $(X, \varrho(\mathcal{G}))$ is an antispaces with a minus topology. (X, \mathcal{G}) and $(X, \varrho(\mathcal{G}))$ determine themselves mutually.

In particular, if (X, \mathcal{G}) is the real line R , then $(X, \varrho(\mathcal{G}))$ is an antispaces and the corresponding topology gives us a compact non-Hausdorff T_1 space, denoted by ϱR , and a large part of mathematics could be based onto ϱR instead of R , since $\varrho^2 R = R$.

Reference

- [1] *J. de Groot*: An isomorphism criterium in general topology (1966). Bull. Am. Math. Soc. 73 (1967).